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Effective potential for the massive sine-Gordon model

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Abstract
The infrared (IR) scaling laws for the massive sine-Gordon model (MSGM) for the number $d = 2$ of spacetime dimensions are determined by means of the functional form of the differential renormalization group (RG) with gliding sharp cut-off k in momentum space in the local potential approximation. This generically non-perturbative method enables one to determine the flow of an arbitrary number of Fourier amplitudes (couplings) of the periodic piece of the potential for any values of the 'inverse temperature' β^2. It is shown that the dimensionless couplings of the periodic piece of the blocked potential exhibit the trivial IR scaling law, $\sim k^{-2}$ due to the mass term breaking periodicity explicitly. Three distinct phases are identified in the space of the bare parameters. The RG flow for phases in which the intrinsic scale k_{SG} of the corresponding phases of the sine-Gordon model (SGM) is larger than the mass, exhibits features similar to those of the RG flow for the SGM, and the effective potentials of these phases appear as IR fixed points.

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1. Introduction

One of the main achievements of the renormalization group (RG) method is the prediction of universality classes, family of theories which may differ on microscopical, ultraviolet (UV) scales but agree on long distances. This feature arises by a simple argument, the classification of the scaling laws, the dependence of the effective parameters of the theories on the observational scale. It is worthwhile noting that such a study is usually carried out in the UV regime but the observations supporting universality are made at long distances, in the infrared (IR) regime. Universality, the convergence of the RG trajectories becomes more robust if the UV scaling regime extends more.

Is this prediction independent of the IR scaling laws? The affirmative answer to this question usually comes from a tacit assumption, namely on the triviality of the IR scaling

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laws around the Gaussian fixed point. By triviality we mean that the only relevant operator is Gaussian, it is actually the mass term. In fact, the fluctuations are suppressed for distances beyond the correlation lengths, the Compton wavelengths of the massive excitations and the scale dependence of the effective interaction strengths slows down at such distances. Since relevant coupling constants around the IR fixed point induce runaway trajectories there is no room for interaction-induced accumulation in effective coupling strengths in the IR regime. The Gaussian operators are exception to this rule. In the case of a traditional kinetic energy whose coefficient is kept invariant by definition the only relevant operator is the mass. Once the RG trajectories have converged during the UV scaling regime they remain close after traversing the crossover located around the correlation length which separate the UV and IR scaling regimes. The only difference which remains from the different microscopical initial conditions is the value of the IR relevant parameter, the mass, in other words an overall scale factor.

Are there other exceptions to the triviality of the IR scaling regime? Certain classically scale invariant models for which the rudimentary argument outlined above is inapplicable may produce non-trivial IR scaling laws. For example, asymptotically free theories must display fundamentally different UV and IR scaling laws even though the explicit, mathematically acceptable demonstration of this property remains to be seen. The goal of this work is to give another class of models where the IR scaling laws differ from the UV ones in a non-trivial manner.

Massless, superrenormalizable theories which are plagued by collinear, IR divergences are known to require a partial resummation of the perturbation series [1], the precursor of major changes in scaling laws. Therefore massless theories indeed do generate non-Gaussian IR fixed point but with non-perturbative scaling laws, beyond the reach of simple studies. For such theories dynamics may generate an intrinsic scale by dimensional transmutation, at which the scaling laws change.

There is however also another possibility of inducing non-trivial IR scaling laws even around the Gaussian fixed point. The starting point is the observation that it is the classical dimension which determines the scaling laws around the Gaussian fixed point in the framework of the loop expansion. The classical dimension of the variable $\partial^m\phi^n$ in d dimensions is $m + n(d-2)/2$ in mass unit. Higher powers of the derivative or higher powers of this expression in an effective vertex lead to faster suppression of the corresponding effective coupling constant for $d > 2$. The gap in the scaling dimensions around its minimal value, corresponding to $m = 0$ and $n = 1$ requires the departure from the vicinity of the Gaussian fixed point in order to establish new, substantially different scaling laws. The straightforward idea is to suppress this gap by considering $d = 2$ and to try to exploit the infinitely degenerated spectrum of the two-dimensional theories in order to cook up non-trivial scaling laws.

We are looking for a principle to lump together infinitely many non-derivative vertices of the type ϕ^n for the number of spacetime dimensions $d = 2$. The simplest argument should come from symmetry reasoning for a local potential $U(\phi)$. For a single component real scalar field the natural symmetry is periodicity,

$$U(\phi + \Delta) = U(\phi), \tag{1}$$

the fundamental symmetry $\pi_1[U(1)]$ of the configuration space, with the constant shift Δ of the field variable. For a non-periodic potential all coupling constants have the same scaling dimension as the mass square in two dimensions. At the same time the sine-Gordon model (SGM) is known to support different scaling laws due to the periodicity and also exhibits an intrinsic scale k_{SG} below which the IR scaling laws appear. We shall therefore study the competition of the scaling laws arising from the non-periodic mass term and a periodic potential in two-dimensional models. In other words, we inspect the competition between the explicit breaking of periodicity and the periodicity induced dynamics generating the intrinsic scale k_{SG}.

The mass term, added to the sine-Gordon Lagrangian breaks the symmetry (1) explicitly. This symmetry breaking should correspond to a phase transition. In order to see this we note that the scaling dimensions of an elementary vertex differ by a finite amount in the aperiodic massive model (−2) and in the periodic SGM. The different speed of the scale dependence induces substantially different effective IR vertices in this manner in the presence or absence of the symmetry (1) even for the approximately same microscopical initial conditions. This is after all the generic mechanism to explain the appearance of phase transitions at the point where formal symmetries are lost or restored dynamically.

Our goal in this work is to map out the phase structure of the SGM and that of the massive sine-Gordon model (MSGM) and to determine the IR scaling laws. Although the confining features of the vacuum of the MSGM have been thoroughly investigated [2, 3], there is a lack of determination of the relevant operators of the low-energy effective theory. It is worthwhile mentioning that the differential RG method in momentum space with the sharp moving cut-off (i) is adequate to determine the effective theories at various energy scales for a given bare theory, and finally the low-energy effective theory for arbitrary values of the parameter β^2 [4–6] and (ii) enables one to treat models with condensates in the vacuum. For the latter the sharp cut-off is crucial, since it allows one to treat differently the modes above and below the condensation scale. Namely, the condensation is signalled by non-trivial saddle points in the path integral for the modes below the condensation scale, the so-called spinodal instability and results in tree-level renormalization [7, 8]. Furthermore, the functional RG method used here enables one to treat the periodic piece of interaction without Taylor expansion, i.e. without violating periodicity and to follow up the RG flow of infinitely many Fourier amplitudes of the periodic potential.

The phase structure of the sine-Gordon-type models are more than pure field-theoretic issues, since e.g. the layered sine-Gordon model [9] is an adequate treatment for the vortex dynamics of the high transition-temperature superconductors, or MSGM is capable of describing the vortices in a two-dimensional (2D) superconductor in the presence of magnetic field [10]. The critical behaviour of vortices in superconducting layers can be treated by the SGM. Due to the magnetic field the logarithmic interaction is valid only up to a length scale at which point it approaches a constant. The dynamically arising intrinsic scale in the system behaves just as a mass scale; thus the proper model to treat this effect is the MSGM. For the first glance the phase structure of the MSGM might seem to be trivial, since all the couplings are relevant in the IR limit due to the finite mass [4]. However, the RG treatment generates infinitely many higher frequency Fourier amplitudes into the self-interaction term. The fact that all the couplings are IR relevant strongly implies that they are not independent. To get a correct picture for the phase structure of the MSGM one has to investigate the effective potential as the function of the bare couplings. Different dependences of the IR couplings on the bare ones and/or the different relations among the IR couplings mean different phases.

This paper is organized as follows. In section 2, the differential RG in momentum space is applied in the local potential approximation (LPA) to the MSGM. In section 3 the UV scaling laws are derived analytically. The numerical methods used to determine the RG flow of the blocked potential are discussed in section 4. Section 5 contains the discussion of our results on the blocked potential in the crossover and the IR scaling regions for the SGM and the MSGM, as well. Finally, our findings are summarized in section 6.

2. Blocked potential

The blocked potential is obtained with the help of the differential RG in momentum space, restricting ourselves to the lowest order of the gradient expansion, the LPA. RG in momentum space is one of the methods which are applicable to the determination of the effective theory, the blocked action at the gliding momentum scale $k \in [0, \Lambda]$ belonging to a given bare theory at the UV cut-off Λ. In this approach the field fluctuations are classified according to their frequencies and the blocking is performed by integrating out the high-frequency modes above the cut-off k. The blocked action S_k at the scale k is defined via

$$\mathrm{e}^{-S_k[\phi]} = \int \mathcal{D}[\phi'] \, \mathrm{e}^{-S_\Lambda[\phi+\phi']} \tag{2}$$

keeping the generating functional invariant,

$$Z = \int \mathcal{D}[\phi] \int \mathcal{D}[\varphi] \, \mathrm{e}^{-S_\Lambda[\phi+\phi']} = \int \mathcal{D}[\phi] \, \mathrm{e}^{-S_k[\phi]}, \tag{3}$$

where S_Λ denotes the bare action, and the field variable is decomposed into the high-frequency modes $\phi'_x = \int_{|p|>k} \phi_p \, \mathrm{e}^{\mathrm{i}px}$ and the low-frequency ones $\phi_x = \int_{0\leqslant|p|\leqslant k} \phi_p \, \mathrm{e}^{\mathrm{i}px}$. In the differential RG in momentum space the high-frequency modes are integrated out step by step in infinitesimal momentum shells of thickness Δk,

$$\mathrm{e}^{-S_{k-\Delta k}[\phi]} = \int \mathcal{D}[\phi'] \, \mathrm{e}^{-S_k[\phi+\phi']}, \tag{4}$$

where the field variable $\phi'_x = \int_{k-\Delta k\leqslant|p|\leqslant k} \phi_p \, \mathrm{e}^{\mathrm{i}px}$ contains only modes with momenta from the infinitesimal momentum shell of radius k. The higher-frequency Fourier modes are splitted further into the sum of the saddle point field configuration ϕ^{cl} and the remaining field fluctuations φ:

$$\phi' = \phi^{cl} + \varphi. \tag{5}$$

In order to evaluate the blocked action in equation (4), we expand the blocked action in Taylor series around its saddle point field configuration

$$S_k[\phi + \phi'] = S_k[\phi + \phi^{cl}] + \frac{1}{2} \int_{p_1,p_2} \varphi_{p_1} S_k^{(2)}(p_1, p_2) \varphi_{p_2} + \mathcal{O}(\varphi^3), \tag{6}$$

where $S_k^{(2)}(p_1, p_2) = \delta^2 S_k / \delta\phi_{p_1} \delta\phi_{p_2}$. It can be shown [6, 11] that the higher order terms in the expansion can be neglected, since they give higher order contributions in $\Delta k/k$. Therefore the functional integral on the rhs of equation (4) becomes Gaussian and can be performed exactly in the limit $\Delta k \to 0$. From equation (6) one arrives at a difference equation for the blocked action

$$\frac{1}{\Delta k}(S_k[\phi + \phi^{cl}] - S_{k-\Delta k}[\phi]) = -\frac{\hbar}{2\Delta k} \mathrm{Tr}' \ln S_k^{(2)}, \tag{7}$$

where the trace Tr' is taken over the modes with momenta in the momentum shell $[k - \Delta k, k]$. One can distinguish two different cases according to the value of the saddle point ϕ^{cl}:

(i) When the saddle point is trivial, i.e. it lies at $\phi'_x = 0$, then equation (7) becomes an integro-differential equation for the blocked action

$$\partial_k S_k = -\lim_{\Delta k\to 0} \frac{\hbar}{2\Delta k} \mathrm{Tr} \ln S_k^{(2)}. \tag{8}$$

We note that the saddle point ϕ^{cl} of the path integral remains trivial as long as the second functional derivative of the blocked action is positive definite.

(ii) When the restoring force for the field fluctuations with momenta from the momentum shell $k - \Delta k < |p| < k$ vanishes, i.e. $S_k^{(2)} = 0$, their amplitudes can grow to finite values, and ϕ^{cl} becomes non-vanishing. This is the so-called spinodal instability. In this case one obtains the tree-level blocking relation [6, 12]

$$S_{k-\Delta k}[\phi] = \min_{\phi'}(S_k[\phi + \phi']) = S_k\big[\phi + \phi_x^{cl}\big], \tag{9}$$

where the minimum is sought over the field configurations ϕ'_x including only Fourier modes with momenta from the infinitesimal momentum shell $k - \Delta k < |p| < k$.

In order to convert the functional differential equation to much simpler coupled differential equations, one has to impose an ansatz for the blocked action. We use the local potential approximation (LPA) for the blocked action which is the leading order of the gradient expansion

$$S_k = \int_x \left[\frac{1}{2}(\partial_\mu \phi_x)^2 + V_k(\phi_x)\right]. \tag{10}$$

Depending on the value of the saddle point ϕ^{cl} of the path integral one can obtain different evolution equations for the blocked potential.

(i) For vanishing saddle point one arrives at the exact Wegner–Houghton RG (WHRG) equation [11] for the blocked potential

$$k\partial_k V_k(\phi) = -\alpha_d k^d \ln\left(\frac{k^2 + V_k''(\phi)}{k^2}\right), \tag{11}$$

where $V_k''(\phi) = \partial_\phi^2 V_k(\phi)$ and $\alpha_d = \Omega_d/2(2\pi)^d$ with Ω_d being the entire solid angle in the d-dimensional momentum space. In order to remove the trivial scale dependence of the coupling constants, the WHRG equation (11) has to be rewritten in terms of dimensionless quantities. For the number of dimensions $d = 2$ the field variable is dimensionless and one finds

$$(2 + k\partial_k)\, \tilde{V}_k(\phi) = -\alpha_2 \ln(1 + \tilde{V}_k''(\phi)), \tag{12}$$

where $\tilde{V}_k = k^{-2} V_k$ has been introduced and $\alpha_2 = 1/4\pi$.

(ii) In the case of non-vanishing saddle point, when we restrict ourselves to the search of the minimum among the plane waves in the spatial direction, equation (9) reduces to the equation

$$\tilde{V}_{k-\Delta k}(\phi) = \min_\rho \left[\rho^2 + \frac{1}{2}\int_{-1}^{1} \mathrm{d}u\, \tilde{V}_k(\phi + 2\rho\cos(\pi u))\right] \tag{13}$$

in LPA [7], where the minimum should be sought for the amplitude ρ of the plane wave.

For the blocked potential of the MSGM we make the particular ansatz

$$V_k(\phi) = \tfrac{1}{2} M_k^2 \phi^2 + U_k(\phi) \tag{14}$$

which contains the periodic potential

$$U_k(\phi) = \sum_{n=1}^{\infty} u_n(k)\cos(n\beta\phi) \tag{15}$$

besides the quadratic mass term. It is easy to show that the RG flow keeps the functional subspace of the blocked potentials defined by the ansatz (14) invariant. If the loop renormalization of equation (11) is applicable, then its right-hand side containing the second derivative $V_k''(\phi)$ is purely periodic. Therefore the blocking keeps the mass $M_k^2 = M^2$ unchanged, affects only the evolution of the periodic potential $U_k(\phi)$ and does not generate any additional non-periodic interaction terms. In the region of spinodal instability equation (13) is valid. The integral in its right-hand side can be rewritten as the sum of terms quadratic and periodic in the field variable ϕ. Therefore the minimization with respect to ρ results in a blocked potential which keeps the form of the ansatz (14). The mass term remains unaltered again. Thus the mass is a relevant parameter at all scales, for which the tree-level scaling law

$$\tilde{M}_k^2 = \tilde{M}_\Lambda^2 \left(\frac{k}{\Lambda}\right)^{-2} \tag{16}$$

holds in the LPA. Both equations (11) and (13) keep the parameter β, i.e. the period of the potential $U_k(\phi)$ unaltered in the LPA.

3. UV scaling

The non-periodic mass term influences the dynamics weakly at high energies and its importance grows gradually as the energy decreases. The first approximate glance into the impact of the mass term on the renormalized trajectory can therefore be obtained by considering the scaling relations in the UV regime, $\Lambda > k \gg M$.

The bare theory is given by the initial conditions for the RG equations imposed at the UV cut-off which is chosen large enough, $\Lambda^2 \gg |V''_\Lambda|$. Therefore, in the asymptotic UV scaling regime $k \approx \Lambda$ the logarithm in the right-hand side of the WHRG equation (11) can be well approximated by the first term of its Taylor series, keeping the leading order correction in the ratio M/k in order to incorporate the effect of the mass term [12, 14]. Apart from an unimportant, field-independent constant term one obtains

$$(2 + k\partial_k)\tilde{U}_k \approx -\frac{\alpha_2 \tilde{U}''_k}{1 + \frac{M^2}{k^2}} \tag{17}$$

for the periodic part of the potential. It is enlightening to rewrite this equation as a set of independent ordinary linear differential equations for the dimensionless couplings,

$$k\partial_k \tilde{u}_n(k) = \left(\frac{\alpha_2 \beta^2 n^2}{1 + \frac{M^2}{k^2}} - 2\right)\tilde{u}_n(k) \tag{18}$$

with $n > 0$. It shows in a clear manner that the mass-dependent correction prevents the right-hand sides from being vanishing for any value of β^2 and n. The solution of this equation is

$$\tilde{u}_n(k) = \tilde{u}_n(\Lambda) \left(\frac{k}{\Lambda}\right)^{-2} \left(\frac{k^2 + M^2}{\Lambda^2 + M^2}\right)^{\beta^2 n^2/8\pi}. \tag{19}$$

4. Numerical methods for the WHRG equations

Even in the LPA the determination of the low-energy effective theory for the MSGM represents a non-trivial numerical problem. As discussed at the end of section 2, the problem reduces to the determination of the RG flow of the periodic potential when the ansatz (14) is used. It has been demonstrated for the SGM [12] that the usual strategy to obtain RG flow equations for the coupling constants based on the Taylor expansion of the blocked potential, violates periodicity and leads to incorrect IR behaviour. Therefore, one has to use the method developed in [12, 14] which retains the symmetry of the periodic part and relies on its Fourier expansion (15) in the internal space.

In order to obtain the coupled set of RG flow equations for the coupling constants of the MSGM it is more convenient to take the first derivative of both sides of equation (12) with respect to the field variable ϕ,

$$(2 + k\partial_k)\tilde{V}_k'(\phi) = -\alpha_2 \frac{\tilde{V}_k'''(\phi)}{1 + \tilde{V}_k''(\phi)}. \tag{20}$$

Equation (20) can be satisfied if and only if the periodic and non-periodic terms on its both sides are identical separately. No evolution arises for the dimensionful mass (cf equation (16)), for the only non-periodic term of the ansatz (14). The periodic part of the blocked potential evolves according to the equation

$$\left[1 + \tilde{M}_k^2 + \tilde{U}_k''(\phi)\right][2 + k\partial_k]\tilde{U}_k'(\phi) = -\alpha_2 \tilde{U}_k'''(\phi). \tag{21}$$

By inserting the dimensionless form of equation (15) into equation (20) we get a coupled set of ordinary differential equations for the dimensionless Fourier amplitudes,

$$\left(1 + \tilde{M}_k^2\right)(2 + k\partial_k)n\tilde{u}_n(k) = \alpha_2\beta^2 n^3 \tilde{u}_n(k) + \frac{\beta^2}{2}\sum_{s=1}^{N} s A_{n,s}(k)(2 + k\partial_k)\tilde{u}_s(k), \tag{22}$$

where $A_{n,s}(k) = (n-s)^2\tilde{u}_{|n-s|} - (n+s)^2\tilde{u}_{n+s}$ and N represents the number of Fourier modes taken into account. These equations have been solved numerically.

The WHRG equation loses its validity at some finite momentum scale $k_c \neq 0$ because the argument of the logarithm vanishes or becomes negative, indicating tree-level instabilities. The RG trajectory should then be followed for $k < k_c$ by the help of the tree-level blocking relation (13) which can be rewritten for the periodic part of the blocked potential as

$$\tilde{U}_{k-\Delta k}(\phi) = \min_{\rho}\left[\left(1 + \tilde{M}_k^2\right)\rho^2 + \sum_{n=1}^{\infty}\tilde{u}_n(k)\cos(n\beta\phi)J_0(2n\beta\rho)\right], \tag{23}$$

where J_0 stands for the zeroth order Bessel function. The minimum of the bracket on the right-hand side was found numerically. The Fourier amplitudes at the lower scale $k - \Delta k$ was determined also numerically by making use of the orthogonality and completeness of the Fourier modes. Let us emphasize again that the period $\Delta = 2\pi/\beta$ in the internal space is kept constant in each step of the tree-level blocking because the expression to be minimized in ρ is periodic in the internal space, together with the position of the minimum, $\rho_k(\phi) = \rho_k(\phi + \Delta)$, and the resulting blocked potential $\tilde{U}_{k-\Delta k}(\phi) = \tilde{U}_{k-\Delta k}(\phi + \Delta)$.

By relying on our experience obtained by solving the similar numerical problem for the SGM, equation (22) was integrated by a C++ code starting from the UV cut-off $\Lambda = 1$ down to the critical value $k_c^2(\phi) = -\partial_\phi^2 V_{k_c}(\phi)$ by means of the fourth-order Runge–Kutta algorithm and the values $\Delta k/k = 10^{-p}$ with $p = 3$ or $p = 4$. There were no appreciable changes in the numerical results by increasing p further. Furthermore we found that the change in the scaling of the first few couplings is negligible by increasing N above 10. In order to determine the scaling of the coupling constants below the critical scale k_c a computer algebra code has been developed using the symbolic programming language Mathematica to solve the tree-level blocking relation (23).

Numerical calculations have been performed for various initial conditions and showed that there is a region in the parameter space $(M^2/\Lambda^2, \beta^2/\pi)$ where spinodal instability occurs along the RG trajectory at some critical scale k_c. No spinodal instability occurs for sufficiently large mass parameter, $M^2 > M_c^2$ but there is a lower and an upper critical value, $\beta_{\mathrm{l}}^2(M^2)$ and $\beta_{\mathrm{u}}^2(M^2)$, respectively when $M^2 < M_c^2$ and the spinodal instability is present (absent) inside (outside) of the interval $\beta_{\mathrm{l}}^2(M^2) < \beta^2 < \beta_{\mathrm{u}}^2(M^2)$.

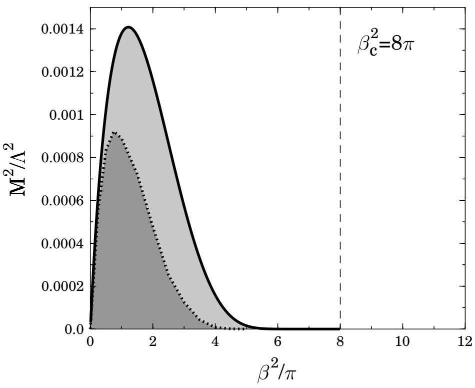

Figure 1. The region of the spinodal instability for the number of Fourier modes $N = 1$ (shaded area) and $N = 10$ (dark area) of the periodic potential, with the choice $u_1(\Lambda) = 0.001$.

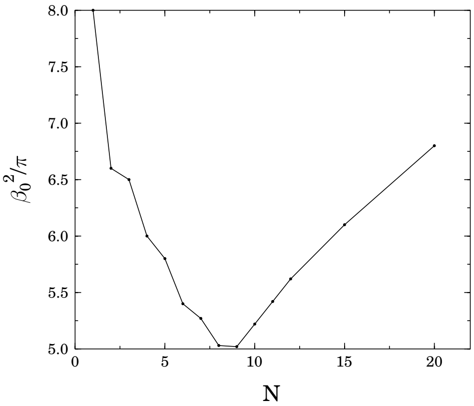

Figure 2. The value of β^2 where the spinodal instability starts in the SGM as the function of the number N of Fourier modes traced numerically during the evolution.

It is instructive to determine the boundary of the spinodal instability for the MSGM restricted to the first Fourier amplitude. This amounts to solving the equation

$$k^2 + M^2 + U_k''(\phi) > 0 \tag{24}$$

together with equation (19) for $n = 1$. The result is

$$k_c^2 = (\Lambda^2 + M^2)\left(\frac{\Lambda^2 + M^2}{\beta^2 u_1(\Lambda)}\right)^{\frac{8\pi}{\beta^2 - 8\pi}} - M^2 \tag{25}$$

when $\beta^2 < 8\pi$ which is depicted in figure 1. But the coupling constant of the first Fourier mode is relevant for $\beta^2 < 8\pi$ according to equation (19) and its growth in the IR regime renders the truncation of the Fourier series unstable. In fact, the phase boundary, obtained numerically with $N = 10$ and shown by dotted line in figure 1 indicates a strong dependence on the truncation which gradually disappears as $\beta \to 0$ due to the factor β^2 on the right-hand side of equation (19).

In order to see better what happens slightly below $\beta^2 = 8\pi$ we show in figure 2 the dependence of the value β^2 where the spinodal instability sets in for the SGM on the order of

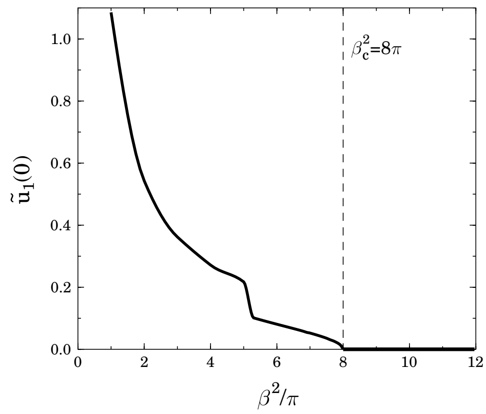

Figure 3. The IR value of the first Fourier amplitude for the sine-Gordon model.

truncation, N. The lesson of this figure is that the Fourier series converges slowly in this phase and one needs rather high order in the truncation to stabilize the results.

The dimensionful effective potential of the SGM is approaching zero for every value of β^2. However one can find differences if one considers the form of the dimensionless potential. The phase structure of the SGM model can be seen clearly by following the β-dependence of the value of the coupling constant $\tilde{u}_1$ at $k = 0$, shown in figure 3. We can distinguish two distinct regions along the β^2 axis.

(i) $\beta^2 > \beta_c^2 = 8\pi$. $\tilde{U}_{k\to 0} \to 0$, the dimensionless couplings scale as

$$\tilde{u}_n(k \to 0) \sim k^{n(\frac{\beta^2}{4\pi}-2)} \tag{26}$$

in the IR limit and the asymptotic behaviour $\tilde{u}_n(k \to 0)$ are given by the first Fourier mode $\tilde{u}_1(k)$ only for any n [15].

(ii) $\beta^2 < \beta_0^2$. The asymptotic IR form of the effective potential becomes independent of the choice of the bare coupling constants $u_n(\Lambda)$, while the IR values $\tilde{u}_n(0)$ do not vanish. The effective potential is expected to take the form

$$\tilde{U}_{k\to 0} = -\tfrac{1}{2}\phi^2, \tag{27}$$

for $\phi \in [-\pi/\beta, \pi/\beta]$ [7], and the IR values $\tilde{u}_n(0)$ can be obtained from the Fourier series of the periodic potential consisting of similar parabola sections set forth along the ϕ-axis periodically. The value of $\tilde{u}_1(0)$ depends strongly on the truncation for $5.2\pi < \beta^2 < 8\pi$ (cf figure 2) and the irregular feature of the solid line in figure 3 in this interval should disappear as $N \to \infty$.

It is worthwhile noting that both in the phase with spinodal instability an intrinsic scale k_{SG} is generated dynamically at which the evolution receives tree-level contributions and below which the couplings $\tilde{u}_n(k)$ are scale independent.

5. The massive sine-Gordon model

The phase boundaries in the MSGM are more involved than those of the SGM due to their mass dependence, witnessed by equation (25) or figure 1. The actual IR scaling laws for the MSGM had to be determined numerically. Calculations for the RG flow have been performed

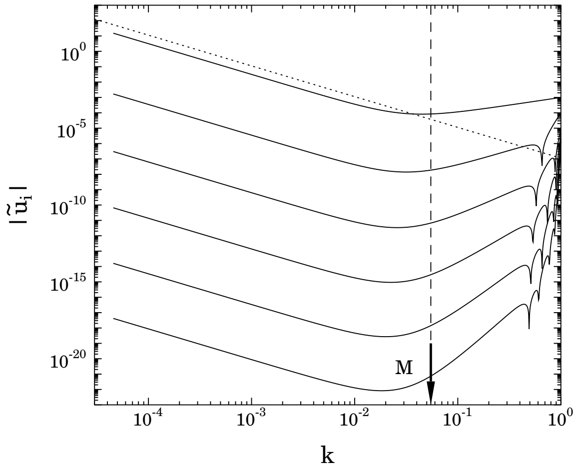

Figure 4. RG flow of the first few Fourier amplitudes of the periodic part of the blocked potential for $\beta^2 = 12\pi$ and $M^2 = 0.003$. The dotted and the dashed lines indicate the scaling law $\sim k^{-2}$ and the scale $k = M$, respectively.

for various values of the bare parameter β^2. Various initial conditions for the bare couplings $\tilde{u}_n(\Lambda)$ were chosen. The magnitude of the dimensionless couplings $|\tilde{u}_n(k)|$ is plotted in figure 4. On the log–log plot the sharp minima of the curves correspond to $\ln|\tilde{u}_n| \to -\infty$ accompanied with the changes of the signs of the couplings $\tilde{u}_n$.

The flow diagrams reveal the following qualitative features:

(i) The asymptotic UV scaling region for $M \ll k$ the coupling constants follow the scaling laws where the mass dependence turns on gradually as k is decreased as in equation (19). In particular, one finds the mass-independent exponents $\alpha_2 n^2\beta^2 - 2$. For example all coupling constants are UV irrelevant in figure 4.

(ii) There is an IR scaling region for $k \ll M$. Quantum fluctuations with momenta $|p| < M$ are frozen out and cannot alter the physics any more. Therefore the tree-level trivial scaling of the dimensionless couplings $|\tilde{u}_n(k)| \sim k^{-2}$ characterizes this region [10, 16, 17]. Correspondingly, a non-vanishing dimensionful periodic potential is left over as $k \to 0$. This should be contrasted with the massless SGM where the dimensionful periodic potential is washed out completely in the IR limit due to the requirement of periodicity and convexity [12].

The scaling $|\tilde{u}_n| \sim k^{-2}$ can be made plausible also by a simple analytic consideration. Namely, one can show that equation (22) rewritten as

$$\sum_s \mathcal{B}_{n,s} k\partial_k \tilde{u}_s(k) = \sum_s \left[-2\mathcal{B}_{n,s} + \frac{\alpha_2\beta^2 n^3 k^2}{k^2 + M^2} \right] \tilde{u}_s(k), \qquad (28)$$

with $\mathcal{B}_{n,s} = s\delta_{n,s} - \frac{1}{2}s\beta^2 k^2 A_{n,s}/(k^2 + M^2)$ supports the assumption $\tilde{u}_n(k) = u_n(0)k^{-2}$. Indeed, one obtains that $\mathcal{B}_{n,s}$ is of order k^0, while the second term in the right-hand side of equation (28) can be neglected for $k \ll M$, so that one recovers

$$\tilde{u}_n(k) \sim u_n(0)k^{-2+\eta_n} \sim u_n(0)k^{-2} \qquad (29)$$

for the IR scaling law.

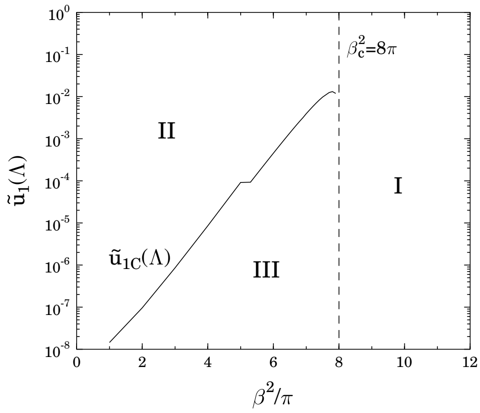

Figure 5. The phase structure of the massive sine-Gordon model.

(iii) The scaling laws in the crossover, between the UV and the IR regimes are determined by the mass M and the intrinsic scale $k_{\rm MSG}$ where the tree-level contributions to the scaling laws set in. When $M < k_{\rm MSG}$ then non-trivial saddle points appear during the evolution for $k < k_{\rm MSG}$ and they become suppressed by the mass around $k \approx M$. If $k_{\rm MSG} < M$ then the tree-level evolution is altogether suppressed and the UV scaling continues directly in the massive IR scaling laws after a short crossover. Below the UV scaling region but above M or $k_{\rm MSG}$ one finds several changes of the signs in the coupling constant $\tilde{u}_n(k)$ for $n \geqslant 2$ due to the nonlinear terms of the RG flow equation.

Although the coupling constants of the MSGM are relevant in the IR limit and their scaling always satisfies $\tilde{u}_n(k) \sim k^{-2}$ for $k \ll M$, it does not necessarily mean that the MSGM exhibits only a single phase. It matters how the IR region is achieved and the phase transitions are identified by investigating the dependences of the IR coupling constants on their bare UV values and/or the relations among the IR coupling constants. The effective potential can be obtained by integrating out all the modes of the scalar field in the generating functional and it depends on the IR limiting values of the coupling constants only. What we have to do is to express the IR limiting values of $\tilde{u}_n(k \to 0)$ in terms of the bare coupling constants and locate phase transitions as singularities in these functions.

Let us now turn to the quantitative details. We restrict our considerations for a small, fixed value of $M^2/\Lambda^2 = 10^{-9}$. As opposed to the SGM one needs now the two-dimensional plane $(\beta^2, \tilde{u}_1(\Lambda))$ to map the phase structure out. There are two lines shown in figure 5 which separate the phases. The dashed line depicts a phase boundary at $\beta_c^2 = 8\pi$ and the solid line shows a phase boundary lying at β^2-dependent critical values $\tilde{u}_{1C}(\Lambda)$. These curves split the plane $(\beta^2, \tilde{u}_1(\Lambda))$ into three regions, distinguished by the different IR behaviour of the dimensionful effective potential of equation (15).

(i) $\beta^2 > 8\pi$. Region I corresponds to the large β^2 phase of the SGM with the modification that the scaling laws are modified for $k < M$. We found no evidence of further phase separations in this region at different values of $\tilde{u}_1(\Lambda)$.

(ii) $\beta^2 < 8\pi$ *and* $\tilde{u}_{1C}(\Lambda) < \tilde{u}_1(\Lambda)$. Region II also inherits the properties of the SGM because the coupling constants are strong enough to generate an intrinsic scale by saddle-point formation at high enough kinetic energy, i.e. $k_{\rm MSG} > M$. The saddle points of the blocking become trivial at lower scales, $k < M$ and the scaling laws

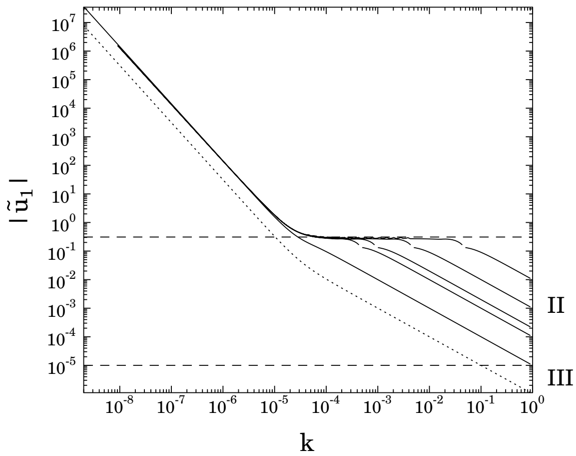

Figure 6. RG flow of the first Fourier amplitude of the periodic part of the blocked potential for several values of $\tilde{u}_1(\Lambda)$, $\beta^2 = 4\pi$ and $M^2 = 10^{-9}$. When the renormalized trajectories reach the horizontal dashed line then spinodal instability sets in and the potential receives universal, $\tilde{u}_1(\Lambda)$-independent contributions.

display the trivial, massive evolution towards the IR end point. The potential of the blocked action receives contributions which are independent from the value of the initial $\tilde{u}_{1C}(\Lambda)$ but the evolution below the mass gap, $k < M$, generates β and M-dependence, $c_n = u_n(k \to 0)/(-u_1(k \to 0))^n = F_n(\beta^2, M)$. The universality can be represented by some RG flows, see figure 6. The solid lines show that flows of the first coupling starting from $\tilde{u}_1(\Lambda)$ value which belongs to region III.

(iii) $\beta^2 < 8\pi$ *and* $\tilde{u}_1(\Lambda) < \tilde{u}_{1C}(\Lambda)$. As $\tilde{u}_1(\Lambda)$ is decreased k_{MSG} reaches the mass scale and the tree-level evolution remains trivial along the full renormalized trajectory. The SGM-type scaling gives rise to the trivial massive IR scaling laws in this region and the effective potential displays dependence on each parameter of the bare action.

In figure 6 the dotted line shows such a flow which avoids the spinodal instability.

The fundamental group symmetry is broken explicitly in the MSGM by the mass term and a non-vanishing dimensionful effective potential is left over in the limit $k \to 0$ independently of presence or absence of spinodal instabilities during the evolution. The effective potential is the sum of the mass term and a non-trivial periodic piece. Our numerical results suggest that the amplitude of the periodic modulation of the second derivative of the potential is sufficiently small as $k \to 0$ to preserve the convexity of the effective potential.

There is no saddle point in the IR regime as $k \to 0$ in the MSGM due to the mass term which breaks explicitly the fundamental group symmetry. Nevertheless the vacuum carries the consequences of the saddle points when they appear within a window of scales, $M < k < k_{\mathrm{MSG}}$. Namely, at the lower end of this interval the coupling constants $\tilde{u}_n(M)$ are independent of $\tilde{u}_1(\Lambda)$ and their further evolution according to the scaling law $\tilde{u}_n(k) = (k/M)^{-2}\tilde{u}_n(M)$ gives rise to a universal dimensionless effective potential in the deep IR region. Then a non-vanishing, universal dimensionful blocked potential survives the IR limit $k \to 0$ with the coupling constants $u_n(0) = u_n(M)$. As opposed to this situation, the effective potential shows dependence on each bare parameter of the action in the absence of tree-level evolution.

6. Summary

Using the differential renormalization group approach in the momentum space with a gliding sharp cut-off we determined the effective potential for the massive sine-Gordon model in the local potential approximation. Although many Fourier amplitudes of the periodic piece of the effective potential were taken into account, our numerical results showed that the effective potential depends at most on the bare value of the first Fourier amplitude $\tilde{u}_1(\Lambda)$. One can distinguish three different regions, phases, on the plane $(\beta^2, \tilde{u}_1(\Lambda))$, separated by singular dependence of the IR values of the coupling constants expressed in terms of the UV parameters of the model. It has been argued that the competition between the explicit breaking of periodicity by the mass term and the periodic dynamics responsible for the intrinsic scale k_{SG} of the corresponding SGM hides behind the rich phase structure. If the intrinsic scale k_{SG} of the SGM exceeds the mass scale M, the effective potential of the MSGM reflects some features of the effective potential of the SGM. A detailed and extensive numerical analysis is in progress in order to clarify further peculiarities of the phases outlined above.

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